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Neighbors Degree Sum Energy of Commuting and Non-Commuting Graphs for Dihedral Groups

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Abstract

The neighbors degree sum (NDS) energy of a graph is determined by the sum of its absolute eigenvalues from its corresponding neighbors degree sum matrix. The non-diagonal entries of $NDS-$matrix are the summation of the degree of two adjacent vertices, or it is zero for non-adjacent vertices, whereas for the diagonal entries are the negative of the square of vertex degree. This study presents the formulas of neighbors degree sum energies of commuting and non-commuting graphs for dihedral groups of order $2n$, D_{2n}, for two cases$-$odd and even n. The results in this paper comply with the well known fact that energy of a graph is neither an odd integer nor a square root of an odd integer.

Keywords: commuting graph, non-commuting graph, dihedral group, neighbors degree sum matrix, the energy of a graph.

1 Introduction

For $n \geq 3$, the non-abelian dihedral group of order $2n$, having the composition as its operation, is a group comprises of motion of the regular n-gon concerning reflection and rotation, denoted by $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ [2]. The center of D_{2n} is either $Z(D_{2n}) = \{e\}$ for odd n, or $Z(D_{2n}) = \left\{e, a^{\frac{n}{2}}\right\}$ for even n. For $a^i \in D_{2n}$, its centralizer is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ and for $a^i b \in D_{2n}$, its centralizer is either $C_{D_{2n}}(a^i b) = \{e, a^i b\}$, if n is odd or $C_{D_{2n}}(a^i b) = \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}$, if n is even.

The set of vertices of both commuting and non-commuting graphs is the set which contains all elements of G, excluding the central elements $Z(G)$, written as $G \backslash Z(G)$. The non-commuting graph of a group G, denoted by Γ_G, is constructed by joining two distinct vertices $v_p, v_q \in G \backslash Z(G)$ with an edge whenever $v_p v_q \neq v_q v_p$ [1]. The complement of Γ_G is the commuting graph of a group G, $\bar{\Gamma}_G$, with two distinct vertices $v_p, v_q \in G \backslash Z(G)$ are adjacent whenever $v_p v_q = v_q v_p$ [5]. In addition, this graph is also related to the results in [6, 21, 23, 22], in which the focus group is symmetric group of order n, and in [17, 18] which deal with the symplectic group, and in [4] which discusses the generalized complement of the commuting graph.

As a matter of fact, Γ_G and $\bar{\Gamma}_G$ can be associated with the adjacency matrix, which is an $n \times n$ matrix $A(\Gamma_G) = [a_{pq}]$ or $A(\bar{\Gamma}_G) = [b_{pq}]$ whose entries a_{pq} or b_{pq} are equal to one if v_p and v_q are adjacent; otherwise, it is zero. The characteristic polynomial of Γ_G (or $\bar{\Gamma}_G$) is defined by $P_{A(\Gamma_G)}(\lambda) = det\left(\lambda I_n - A(\Gamma_G)\right)$ (or $P_{A(\bar{\Gamma}_G)}(\lambda) = det\left(\lambda I_n - A(\bar{\Gamma}_G)\right)$, where I_n is an $n \times n$ identity matrix. The roots of $P_{A(\Gamma_G)}(\lambda) = 0$ (or $P_{A(\bar{\Gamma}_G)}(\lambda) = 0$) are known as the eigenvalues of Γ_G (or $\bar{\Gamma}_G$), denoted as $\lambda_1, \lambda_2, \ldots, \lambda_n$. The spectrum of Γ_G (or $\bar{\Gamma}_G$), denoted by $Spec(\Gamma_G)$ (or $Spec(\bar{\Gamma}_G)$) is the list of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, where $m \leq n$, together with their respective multiplicities $k_1, k_2, \ldots, k_m$, written by $\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m}\right\}$.

The energy of Γ_G (or $\bar{\Gamma}_G$) is the sum of all $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$ of Γ_G (or $\bar{\Gamma}_G$) or $\sum_{i=1}^{n} |\lambda_i|$. Gutman pioneered this definition in 1978 [13] and subsequently applied it in the Chemistry field to estimate the property of molecules regarding the π-electron energy. In this case, a molecule is viewed as a graph, with carbon atoms as vertices and hydrogen bonds between carbon atoms as edges. It should be noted that the adjacency energy is never an odd number [3] nor the square root of an odd number [24].

We define the spectral radius of Γ_G as $\rho(\Gamma_G) = max\left\{\lambda | \lambda \in Spec(\Gamma_G)\right\}$. In other words, $\rho(\Gamma_G)$ is a non-negative real number with a center at the origin of the complex plane and is the smallest disc radius containing all the eigenvalues of Γ_G [15]. A study of the spectra radius problem of several graphs has been conducted. A discussion on a spectral radius for the power graph for dihedral groups can be found in [8], whereas [11] describes the signless Laplacian energy and spectral radius for a directed graph. In addition, [14] presents the spectral distance of the hypercube and line graphs.

There has been significant development in algebraic graph theory with regard to commuting and non-commuting graphs over the years. As can be seen in [20, 28, 26, 27], which provide detailed description on the spectral and energy problem of commuting and non-commuting graphs especially for dihedral groups using the spectrum of adjacency, degree sum, degree exponent sum, maximum degree, and minimum degree matrices associated with Γ_G and $\bar{\Gamma}_G$. Similarly, the spectrum associated with an adjacency matrix for commuting graphs for non-abelian finite groups can be found in [9]. In addition, [10] explores broadly on the ordinary spectrum and energy of Γ_G for finite groups, including dihedral groups.

Research has been conducted in graph energies to a significant extent over the past few decades, especially those related to matrices of graphs in which their entries are associated with the number of vertices adjacent to a vertex v_p or simply called as the degree of that vertex, denoted by d_{v_p}. One of them is an $n \times n$ matrix called neighbors degree sum (NDS) matrix introduced by Boregowda and Jummannaver [16]. If we represent Γ_G or $\bar{\Gamma}_G$ using $NDS-$matrix, then $NDS(\Gamma_G) = [nds_{pq}]$ or $NDS(\bar{\Gamma}_G) = [nds_{pq}]$ whose $(p,q)-$th entry is

$$nds_{pq} = \begin{cases} -d_{v_p}^2, & \text{if } v_p = v_q \\ d_{v_p} + d_{v_q}, & \text{if } v_p \text{ and } v_q \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}.$$

However, to the best of our knowledge, limited articles discuss on the energy of Γ_G and $\bar{\Gamma}_G$ of the dihedral groups using NDS eigenvalues. Therefore, the research objective of this study is to present a complete formula of neighbors degree sum energy of Γ_G and $\bar{\Gamma}_G$ for D_{2n}, $n \geq 3$.

2 Preliminaries

The following are some fundamental results that are used in this work to formulate the characteristic polynomials of Γ_G and $\bar{\Gamma}_G$.

Lemma 2.1. [25] *If w, x, y and z are real numbers, and I_n be the $n \times n$ identity matrix and J_n be the $n \times n$ matrix whose all entries are equal to 1, then the determinant of the $(n_1+n_2)\times(n_1+n_2)$ matrix of the form*

$$\begin{vmatrix} (\lambda+w)I_{n_1} - wJ_{n_1} & -yJ_{n_1\times n_2} \\ -zJ_{n_2\times n_1} & (\lambda+x)I_{n_2} - xJ_{n_2} \end{vmatrix}$$

can be simplified in an expression as

$$(\lambda+w)^{n_1-1}(\lambda+x)^{n_2-1}\left(\left(\lambda-(n_1-1)\,w\right)\left(\lambda-(n_2-1)\,x\right)-n_1n_2yz\right),$$

where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 2.1. [12] *If a square matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be partitioned into four blocks A, B, C, D, where $|A| \neq 0$, then*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A|\left|D - CA^{-1}B\right|.$$

Moreover, we also use row and column operations to formulate the determinants of the characteristic polynomial of Γ_G and $\bar{\Gamma}_G$. Therefore, we define the following notations: (i) R_i is the $i-$th row; (ii) $R_i^{'}$ is the new $i-$th row obtained from a row operation; (iii) C_i is the $i-$th column; and (iv) $C_i^{'}$ is the new $i-$th column obtained from a column operation of the characteristic polynomial of Γ_G and $\bar{\Gamma}_G$.

On the other hand, the following are some underlying results focusing on the degree of vertices of Γ_G and $\bar{\Gamma}_G$ for $G = D_{2n}\backslash Z(D_{2n})$ where D_{2n} is the dihedral groups of order $2n$ and $Z(D_{2n})$ is its center.

Theorem 2.2. [19] *Let Γ_G be the non-commuting graph on G, where $G = D_{2n}\backslash Z(D_{2n})$. Then*

1. *the degree of* a^i *on* Γ_G *is* $d_{a^i} = n$*, and*

2. *the degree of* $a^i b$ *on* Γ_G *is* $d_{a^i b} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.3. [28] *Let* $\bar{\Gamma}_G$ *be the commuting graph for* G*, where* $G = D_{2n} \backslash Z(D_{2n})$*. Then*

1. *the degree of* a^i *on* $\bar{\Gamma}_G$ *is* $d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd} \\ n-3, & \text{if } n \text{ is even,} \end{cases}$ *and*

2. *the degree of* $a^i b$ *on* $\bar{\Gamma}_G$ *is* $d_{a^i b} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

3 Main Results

This section presents the results on the energy of Γ_G and $\bar{\Gamma}_G$ for dihedral groups using the corresponding neighbors degree sum matrix. For $n \geq 3$, we consider two cases−1) odd n and 2) even n, because of the difference in the center properties of the dihedral group D_{2n}.

3.1 Neighbors Degree Sum Energy of Non-Commuting Graph for Dihedral Groups

Theorem 3.1. *Let* Γ_G *be the non-commuting graph on* $G = D_{2n} \backslash Z(D_{2n})$*, where* $n \geq 3$*. Then the neighbors degree sum energy for* Γ_G *is*

$$E_{NDS}(\Gamma_G) = \begin{cases} (n-2)^2 n^2 + 4n(n-1)^2 + \sqrt{n^4 + 4n(n-1)(3n-2)^2}, & \text{for odd } n \\ (n-3)^2 n^2 + 4n(n-2)^2 + \sqrt{n^4 + 4n(n-2)(3n-4)^2}, & \text{for even } n. \end{cases}$$

Proof. 1. When n is odd, considering the definition of the neighbors degree sum matrix together with the centralizer of each element in D_{2n} and the properties from Theorem 2.2, then $NDS(\Gamma_G)$ is a $(2n-1) \times (2n-1)$ matrix as follows:

$$NDS(\Gamma_G) = \begin{bmatrix} -n^2 I_{n-1} & (3n-2) J_{(n-1)\times n} \\ (3n-2) J_{n \times (n-1)} & -\left(4(n-1)^2 + 4(n-1)\right) I_n + 4(n-1) J_n \end{bmatrix}.$$

Here, the characteristic polynomial of $NDS(\Gamma_G)$ can be written by

$$P_{NDS(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + n^2) I_{n-1} & -(3n-2) J_{(n-1)\times n} \\ -(3n-2) J_{n \times (n-1)} & \left(\lambda + 4(n-1)^2 + 4(n-1)\right) I_n - 4(n-1) J_n \end{vmatrix}. \quad (1)$$

In order to determine the roots of $P_{NDS(\Gamma_G)}(\lambda) = 0$, elementary row and column operations on $P_{NDS(\Gamma_G)}(\lambda)$ need to be performed.

Step 1: For every $1 \leq i \leq n-1$, we substitute R_{n+i} by $R'_{n+i} = R_{n+i} - R_n$. Then we see that Equation (1) is

$$\begin{vmatrix} (\lambda + n^2) I_{n-1} & -(3n-2) J_{(n-1)\times 1} & -(3n-2) J_{(n-1)\times(n-1)} \\ -(3n-2) J_{1 \times (n-1)} & \lambda + 4(n-1)^2 & -4(n-1) J_{1 \times (n-1)} \\ 0_{(n-1)} & -\left(\lambda + 4(n-1)^2 + 4(n-1)\right) J_{(n-1)\times 1} & \left(\lambda + 4(n-1)^2 + 4(n-1)\right) I_{n-1} \end{vmatrix}. \quad (2)$$

Step 2: We replace C_n by $C_n^{'} = C_n + C_{n+1} + C_{n+2} + \ldots + C_{2n-1}$, then we deduce that Equation (2) is

$$\begin{vmatrix} (\lambda+n^2)I_{n-1} & -n(3n-2)J_{(n-1)\times 1} & -(3n-2)J_{(n-1)\times(n-1)} \\ -(3n-2)J_{1\times(n-1)} & \lambda & -4(n-1)J_{1\times(n-1)} \\ 0_{n-1} & 0_{(n-1)\times 1} & \left(\lambda+4(n-1)^2+4(n-1)\right) I_{n-1} \end{vmatrix}. \tag{3}$$

Step 3: Using Theorem 2.1 with
$A = \begin{bmatrix} (\lambda+n^2)I_{n-1} & -n(3n-2)J_{(n-1)\times 1} \\ -(3n-2)J_{1\times(n-1)} & \lambda \end{bmatrix}$, $B = \begin{bmatrix} -(3n-2)J_{(n-1)\times(n-1)} \\ -4(n-1)J_{1\times(n-1)} \end{bmatrix}$, $C = 0_{(n-1)\times n}$, and $D = \left(\lambda+4(n-1)^2+4(n-1)\right) I_{n-1}$, then Equation (3) is the form of

$$P_{NDS(\Gamma_G)}(\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|\,|D|\,. \tag{4}$$

Now we calculate the first determinant $|A|$ with the next steps.

Step 4: We replace C_i by $C_i^{'} = C_i - C_{n-1}$, for all $1 \leq i \leq n-2$. Then

$$|A| = \begin{vmatrix} (\lambda+n^2)I_{n-2} & 0_{(n-2)\times 1} & -n(3n-2)J_{(n-2)\times 1} \\ -(\lambda+n^2)J_{1\times(n-2)} & \lambda+n^2 & -n(3n-2) \\ 0_{1\times(n-2)} & -(3n-2) & \lambda \end{vmatrix}. \tag{5}$$

Step 5: Replace R_{n-1} by $R_{n-1}^{'} = R_{n-1} + R_1 + R_2 + \ldots + R_{n-2}$, then Equation (5) can be written as

$$|A| = \begin{vmatrix} (\lambda+n^2)I_{n-2} & 0_{(n-2)\times 1} & -n(3n-2)J_{(n-2)\times 1} \\ 0_{1\times(n-2)} & \lambda+n^2 & -n(n-1)(3n-2) \\ 0_{1\times(n-2)} & -(3n-2) & \lambda \end{vmatrix}. \tag{6}$$

Step 6: Again, by Theorem 2.1, we can rewrite Equation (6) as the following:

$$\begin{aligned} |A| &= \left|(\lambda+n^2)I_{n-2}\right| \begin{vmatrix} \lambda+n^2 & -n(n-1)(3n-2) \\ -(3n-2) & \lambda \end{vmatrix} \\ &= \left(\lambda+n^2\right)^{n-2} \left(\lambda^2+n^2\lambda-n(n-1)(3n-2)^2\right). \end{aligned} \tag{7}$$

Meanwhile, since $|D|$ is a diagonal matrix, then the immediate $|D|$ is

$$|D| = \left(\lambda+4(n-1)^2+4(n-1)\right)^{n-1} = \left(\lambda+4n(n-1)\right)^{n-1}. \tag{8}$$

Therefore, if we go back to Equation (4), $P_{NDS(\Gamma_G)}(\lambda)$ is the product of Equations (7) and (8) as the following

$$P_{NDS(\Gamma_G)}(\lambda) = \left(\lambda+n^2\right)^{n-2} \left(\lambda^2+n^2\lambda-n(n-1)(3n-2)^2\right) \left(\lambda+4n(n-1)\right)^{n-1}.$$

Hence, we get the spectrum of Γ_G,

$$\begin{aligned} Spec(\Gamma_G) = \Bigg\{ &\left(\frac{-n^2}{2} + \frac{\sqrt{n^4+4n(n-1)(3n-2)^2}}{2}\right)^1, \left(-4n(n-1)\right)^{n-1}, \\ &\left(-n^2\right)^{n-2}, \left(\frac{-n^2}{2} - \frac{\sqrt{n^4+4n(n-1)(3n-2)^2}}{2}\right)^1 \Bigg\}, \end{aligned}$$

and finally we see that

$$E_{NDS}(\Gamma_G) = (n-2)^2n^2 + 4n(n-1)^2 + \sqrt{n^4+4n(n-1)(3n-2)^2}.$$

2. By Theorem 2.2 for even n, the neighbors degree sum matrix of Γ_G, $NDS(\Gamma_G)$ is an $(2n-2)\times(2n-2)$ matrix as follows:

$$\begin{bmatrix} -n^2 I_{n-2} & (3n-4)J_{(n-2)\times\frac{n}{2}} & (3n-4)J_{(n-2)\times\frac{n}{2}} \\ (3n-4)J_{\frac{n}{2}\times(n-2)} & -\left(4(n-2)^2+4(n-2)\right)I_{\frac{n}{2}}+4(n-2)J_{\frac{n}{2}} & 4(n-2)(J-I)_{\frac{n}{2}} \\ (3n-4)J_{\frac{n}{2}\times(n-2)} & 4(n-2)(J-I)_{\frac{n}{2}} & -\left(4(n-2)^2+4(n-2)\right)I_{\frac{n}{2}}+4(n-2)J_{\frac{n}{2}} \end{bmatrix}.$$

Here, $P_{NDS(\Gamma_G)}(\lambda)$ is

$$\begin{vmatrix} (\lambda+n^2)I_{n-2} & -(3n-4)J_{(n-2)\times\frac{n}{2}} & -(3n-4)J_{(n-2)\times\frac{n}{2}} \\ -(3n-4)J_{\frac{n}{2}\times(n-2)} & \left(\lambda+4(n-2)^2+4(n-2)\right)I_{\frac{n}{2}}-4(n-2)J_{\frac{n}{2}} & -4(n-2)(J-I)_{\frac{n}{2}} \\ -(3n-4)J_{\frac{n}{2}\times(n-2)} & -4(n-2)(J-I)_{\frac{n}{2}} & \left(\lambda+4(n-2)^2+4(n-2)\right)I_{\frac{n}{2}}-4(n-2)J_{\frac{n}{2}} \end{vmatrix}. \tag{9}$$

In order to determine λ, elementary row and column operations on $P_{NDS(\Gamma_G)}(\lambda)$ need to be performed

Step 1: For every $1\le i\le\frac{n}{2}$, we replace $R_{n-2+\frac{n}{2}+i}$ with the new row $R'_{n-2+\frac{n}{2}+i}=R_{n-2+\frac{n}{2}+i}-R_{n-2+i}$. Then we see that Equation (9) is

$$\begin{vmatrix} (\lambda+n^2)I_{n-2} & -(3n-4)J_{(n-2)\times\frac{n}{2}} & -(3n-4)J_{(n-2)\times\frac{n}{2}} \\ -(3n-4)J_{\frac{n}{2}\times(n-2)} & (\lambda+4(n-2)^2+4(n-2))\,I_{\frac{n}{2}}-4(n-2)J_{\frac{n}{2}} & -4(n-2)(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -(\lambda+4(n-2)^2)I_{\frac{n}{2}} & (\lambda+4(n-2)^2)I_{\frac{n}{2}} \end{vmatrix}. \tag{10}$$

Step 2: We replace C_{n-2+i} by the new column $C'_{n-2+i}=C_{n-2+i}+C_{n-2+\frac{n}{2}+i}$, for every $1\le i\le\frac{n}{2}$, then Equation (10) can be stated as

$$\begin{vmatrix} (\lambda+n^2)I_{n-2} & -2(3n-4)J_{(n-2)\times\frac{n}{2}} & -(3n-4)J_{(n-2)\times\frac{n}{2}} \\ -(3n-4)J_{\frac{n}{2}\times(n-2)} & (\lambda+4(n-2)^2+8(n-2))\,I_{\frac{n}{2}}-8(n-2)J_{\frac{n}{2}} & -4(n-2)(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & (\lambda+4(n-2)^2)I_{\frac{n}{2}} \end{vmatrix}. \tag{11}$$

Step 3: According to Theorem 2.1 with
$A=\begin{bmatrix} (\lambda+n^2)I_{n-2} & -2(3n-4)J_{(n-2)\times\frac{n}{2}} \\ -(3n-4)J_{\frac{n}{2}\times(n-2)} & \left(\lambda+4(n-2)^2+8(n-2)\right)I_{\frac{n}{2}}-8(n-2)J_{\frac{n}{2}} \end{bmatrix}$,
$B=\begin{bmatrix} -(3n-4)J_{(n-2)\times\frac{n}{2}} \\ -4(n-2)(J-I)_{\frac{n}{2}} \end{bmatrix}$, $C=0_{\frac{n}{2}\times(n-2+\frac{n}{2})}$, and $D=(\lambda+4(n-2)^2)I_{\frac{n}{2}}$, then Equation (11) is the form of

$$P_{NDS(\Gamma_G)}(\lambda)=\begin{bmatrix} A & B \\ C & D \end{bmatrix}=|A|\,|D|\,. \tag{12}$$

Now we calculate the first determinant, $|A|$ with the next steps:

Step 4: For every $1\le i\le\frac{n}{2}-1$, replace R_{n-1+i} with the new row $R'_{n-1+i}=R_{n-1+i}-R_{n-1}$. Then we see that

$$|A|=\begin{vmatrix} (\lambda+n^2)I_{n-2} & -2(3n-4)J_{(n-2)\times1} & -2(3n-4)J_{(n-2)\times(\frac{n}{2}-1)} \\ -(3n-4)J_{1\times(n-2)} & \lambda+4(n-2)^2 & -8(n-2)J_{1\times(\frac{n}{2}-1)} \\ 0_{(\frac{n}{2}-1)\times(n-2)} & -\left(\lambda+4(n-2)^2+8(n-2)\right)J_{(\frac{n}{2}-1)\times1} & \left(\lambda+4(n-2)^2+8(n-2)\right)I_{\frac{n}{2}-1} \end{vmatrix}. \tag{13}$$

Step 5: We replace C_{n-1} by the new column $C'_{n-1}=C_{n-1}+C_n+C_{n+1}+\ldots+C_{n-2+\frac{n}{2}}$, then Equation (13) can be expressed as

$$|A|=\begin{vmatrix} (\lambda+n^2)I_{n-2} & -n(3n-4)J_{(n-2)\times1} & -2(3n-4)J_{(n-2)\times(\frac{n}{2}-1)} \\ -(3n-4)J_{1\times(n-2)} & \lambda & -8(n-2)J_{1\times(\frac{n}{2}-1)} \\ 0_{(\frac{n}{2}-1)\times(n-2)} & 0_{(\frac{n}{2}-1)\times1} & (\lambda+4(n-2)^2+8(n-2))\,I_{\frac{n}{2}-1} \end{vmatrix}. \tag{14}$$

Step 6: According to Theorem 2.1, then we can express Equation (14) as

$$|A|=|E|\,|F|\,, \tag{15}$$

with $|E| = \begin{vmatrix} (\lambda+n^2)I_{n-2} & -n(3n-4)J_{(n-2)\times 1} \\ -(3n-4)J_{1\times(n-2)} & \lambda \end{vmatrix}$ and
$|F| = \left|\left(\lambda+4(n-2)^2+8(n-2)\right) I_{\frac{n}{2}-1}\right|$.

Step 7: Now we calculate the first determinant, $|E|$, by replacing C_i with $C_i^{'} = C_i - C_{n-2}$, for all $1 \leq i \leq n-3$. Then

$$|E| = \begin{vmatrix} (\lambda+n^2)I_{n-3} & 0_{(n-3)\times 1} & -n(3n-4)J_{(n-3)\times 1} \\ -(\lambda+n^2)J_{1\times(n-3)} & \lambda+n^2 & -n(3n-4) \\ 0_{1\times(n-3)} & -(3n-4) & \lambda \end{vmatrix}. \tag{16}$$

Step 8: Now replace R_{n-2} by $R_{n-2}^{'} = R_{n-2} + R_1 + R_2 + \ldots + R_{n-3}$, then Equation (16) can be written by

$$|E| = \begin{vmatrix} (\lambda+n^2)I_{n-3} & 0_{(n-3)\times 1} & -n(3n-4)J_{(n-3)\times 1} \\ 0_{1\times(n-3)} & \lambda+n^2 & -n(n-2)(3n-4) \\ 0_{1\times(n-3)} & -(3n-4) & \lambda \end{vmatrix}. \tag{17}$$

Step 9: Again, by Theorem 2.1, then Equation (17) can be expressed as the the following:

$$\begin{aligned} |E| &= \left|(\lambda+n^2)I_{n-3}\right| \begin{vmatrix} \lambda+n^2 & -n(n-2)(3n-4) \\ -(3n-4) & \lambda \end{vmatrix} \\ &= \left(\lambda+n^2\right)^{n-4}\left(\lambda^2+n^2\lambda-n(n-2)(3n-4)^2\right). \end{aligned} \tag{18}$$

Step 10: Since $|F|$ is a diagonal matrix, immediately $|F|$ is the product of the main diagonal entries as follows:

$$|F| = \left(\lambda+4(n-1)^2+8(n-2)\right)^{\frac{n}{2}-1} = \left(\lambda+4n(n-2)\right)^{\frac{n}{2}-1}. \tag{19}$$

If we go back to Equation (15), then using Equations (18) and (19), we can see $|A|$ as follows:

$$|A| = (\lambda+n^2)^{n-3}(\lambda^2+n^2\lambda-n(n-2)(3n-4)^2)(\lambda+4n(n-2))^{\frac{n}{2}-1}. \tag{20}$$

Meanwhile, since $|D|$ is a diagonal matrix, then

$$|D| = (\lambda+4(n-2)^2)^{\frac{n}{2}}. \tag{21}$$

Finally, following Equations (20) and (21), we see Equation (12) as the following expression:

$$P_{NDS(\Gamma_G)}(\lambda) = (\lambda+n^2)^{n-3}(\lambda^2+n^2\lambda-n(n-2)(3n-4)^2)(\lambda+4n(n-2))^{\frac{n}{2}-1}(\lambda+4(n-2)^2)^{\frac{n}{2}}.$$

The spectrum of Γ_G is

$$\begin{aligned} Spec(\Gamma_G) = \Bigg\{ &\left(\frac{-n^2}{2}+\frac{\sqrt{n^4+4n(n-2)(3n-4)^2}}{2}\right)^1, (-4n(n-2))^{\frac{n}{2}-1}, (-n^2)^{n-3}, \\ &(-4(n-2)^2)^{\frac{n}{2}}, \left(\frac{-n^2}{2}-\frac{\sqrt{n^4+4n(n-2)(3n-4)^2}}{2}\right)^1 \Bigg\}. \end{aligned}$$

Therefore, the $NDS-$energy of Γ_G is

$$E_{NDS}(\Gamma_G) = (n-3)^2n^2 + 4n(n-2)^2 + \sqrt{n^4+4n(n-2)(3n-4)^2}.$$

□

3.2 Neighbors Degree Sum Energy of Commuting Graph for Dihedral Groups

Theorem 3.2. *Let $\bar{\Gamma}_G$ be the commuting graph on G, where $G = D_{2n} \backslash Z(D_{2n})$, where $n \geq 3$. Then the neighbors degree sum NDS energy for $\bar{\Gamma}_G$ is*

$$E_{NDS}(\bar{\Gamma}_G) = \begin{cases} (n+1)(n-2)^2, & \text{for odd } n \\ n(n-3)^2 + 2n, & \text{for even } n. \end{cases}$$

Proof. 1. When n is odd, considering the definition of the neighbors degree sum matrix together with the centralizer of each element in D_{2n} and the properties from Theorem 2.3, then $NDS(\bar{\Gamma}_G)$ is an $(2n-1) \times (2n-1)$ matrix as follows:

$$NDS(\bar{\Gamma}_G) = \begin{bmatrix} -((n-2)^2 + 2(n-2))I_{n-1} + 2(n-2)J_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

We then obtain the characteristic polynomial of $NDS(\bar{\Gamma}_G)$ as given below:

$$P_{NDS(\bar{\Gamma}_G)}(\lambda) = \begin{vmatrix} (\lambda + (n-2)^2 + 2(n-2))I_{n-1} - 2(n-2)J_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & \lambda I_n \end{vmatrix}. \quad (22)$$

By using Theorem 2.1 with $A = (\lambda + (n-2)^2 + 2(n-2))I_{n-1} - 2(n-2)J_{n-1}$, $B = 0_{(n-1)\times n}$, $C = 0_{n\times(n-1)}$, $D = \lambda I_n$, then Equation (22) can be expressed as

$$P_{NDS(\bar{\Gamma}_G)}(\lambda) = |A|\,|D|\,. \quad (23)$$

It is clear that

$$|D| = \lambda^n. \quad (24)$$

Now see consider $|A|$ and use the following steps of row and column operations:

Step 1: For every $2 \leq i \leq n-1$, replace R_i by $R_i^{'} = R_i - R_1$. Then we see that

$$|A| = \begin{vmatrix} \lambda + (n-2)^2 & -2(n-2)J_{1\times(n-2)} \\ -(\lambda + (n-2)^2 + 2(n-2))J_{(n-2)\times 1} & (\lambda + (n-2)^2 + 2(n-2))I_{n-2} \end{vmatrix}. \quad (25)$$

Step 2: We replace C_1 by $C_1^{'} = C_1 + C_2 + C_3 + \ldots + C_{n-1}$, then we deduce that Equation (25) is an upper triangular matrix

$$|A| = \begin{vmatrix} \lambda + (n-2)^2 & -2(n-2)J_{1\times(n-2)} \\ 0_{(n-2)\times 1} & (\lambda + (n-2)^2 + 2(n-2))I_{n-2} \end{vmatrix}. \quad (26)$$

Thus, $|A|$ is the product of the main diagonal entries of Equation (26) as the following:

$$|A| = (\lambda - (n-2)^2)(\lambda + n(n-2))^{n-2}. \quad (27)$$

From Equations (24) and (27), then our desired equation in (23) is

$$P_{NDS(\bar{\Gamma}_G)}(\lambda) = \lambda^n(\lambda - (n-2)^2)(\lambda + n(n-2))^{n-2}.$$

Hence, the spectrum of $\bar{\Gamma}_G$ is

$$Spec(\bar{\Gamma}_G)) = \left\{ \left((n-2)^2\right)^1, (0)^n, (-n(n-2))^{n-2} \right\},$$

and finally, we see that

$$E_{NDS}(\bar{\Gamma}_G) = (n+1)(n-2)^2.$$

2. By Theorem 2.3 for the even n, then $NDS(\bar{\Gamma}_G)$ is an $(2n-2)\times(2n-2)$ matrix as the following:

$$NDS(\bar{\Gamma}_G) = \begin{bmatrix} -((n-3)^2+2(n-3))I_{n-2}+2(n-3)J_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -I_{\frac{n}{2}} & 2I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 2I_{\frac{n}{2}} & -I_{\frac{n}{2}} \end{bmatrix}.$$

We then get the characteristic polynomial of $NDS(\bar{\Gamma}_G)$ as follows:

$$P_{NDS(\bar{\Gamma}_G)}(\lambda) = \begin{vmatrix} (\lambda+(n-3)^2+2(n-3))I_{n-2}-2(n-3)J_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & (\lambda+1)I_{\frac{n}{2}} & -2I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & -2I_{\frac{n}{2}} & (\lambda+1)I_{\frac{n}{2}} \end{vmatrix}. \tag{28}$$

By using Theorem 2.1 with $A=(\lambda+(n-3)^2+2(n-3))I_{n-2}-2(n-3)J_{n-2}$, $B=0_{(n-2)\times n}$, $C=0_{n\times(n-2)}$, and $D=\begin{bmatrix} (\lambda+1)I_{\frac{n}{2}} & -2I_{\frac{n}{2}} \\ -2I_{\frac{n}{2}} & (\lambda+1)I_{\frac{n}{2}} \end{bmatrix}$, then Equation (28) is the form of

$$P_{NDS(\bar{\Gamma}_G)}(\lambda) = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D|. \tag{29}$$

Now we consider $|A|$ with the following steps:

Step 1: We replace R_i by $R_i^{'}=R_i-R_1$, for every $2\le i\le n-2$. Then we see that

$$|A| = \begin{vmatrix} \lambda+(n-3)^2 & -2(n-3)J_{1\times(n-3)} \\ -(\lambda+(n-3)^2+2(n-3))J_{(n-3)\times1} & (\lambda+(n-3)^2+2(n-3))I_{n-3} \end{vmatrix}. \tag{30}$$

Step 2: We replace C_1 by $C_1^{'}=C_1+C_2+\ldots+C_{n-2}$, then we deduce that Equation (30) is an upper triangular matrix

$$|A| = \begin{vmatrix} \lambda+(n-3)^2 & -2(n-3)J_{1\times(n-3)} \\ 0_{(n-3)\times1} & (\lambda+(n-3)^2+2(n-3))I_{n-3} \end{vmatrix}. \tag{31}$$

Thus, $|A|$ is the product of the main diagonal entries of Equation (31) as the following:

$$|A| = (\lambda-(n-3)^2)(\lambda+(n-1)(n-3))^{n-3}. \tag{32}$$

Meanwhile, by replacing $R_{\frac{n}{2}+i}$ by $R^{'}_{\frac{n}{2}+i}=R_{\frac{n}{2}+i}-R_i$, for every $1\le i\le\frac{n}{2}$ in $|D|$, then

$$|D| = \begin{vmatrix} (\lambda+1)I_{\frac{n}{2}} & -2I_{\frac{n}{2}} \\ -(\lambda+3)I_{\frac{n}{2}} & (\lambda+3)I_{\frac{n}{2}} \end{vmatrix}, \tag{33}$$

and following by replacing C_i by $C_i^{'}=C_i+C_{\frac{n}{2}+i}$, for every $1\le i\le\frac{n}{2}$ in Equation (33), then

$$|D| = \begin{vmatrix} (\lambda-1)I_{\frac{n}{2}} & -2I_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda+3)I_{\frac{n}{2}} \end{vmatrix} = (\lambda-1)^{\frac{n}{2}}(\lambda+3)^{\frac{n}{2}}. \tag{34}$$

From Equations (32) and (34), then our required result in Equation (29) is

$$P_{NDS(\bar{\Gamma}_G)}(\lambda) = (\lambda-(n-3)^2)(\lambda+(n-1)(n-3))^{n-3}(\lambda-1)^{\frac{n}{2}}(\lambda+3)^{\frac{n}{2}}.$$

Therefore, the immediate spectrum and $NDS-$energy of $\bar{\Gamma}_G$ are

$$Spec(\bar{\Gamma}_G) = \left\{((n-3)^2)^1, (1)^{\frac{n}{2}}, (-3)^{\frac{n}{2}}, (-(n-1)(n-3))^{n-3}\right\},$$
$$E_{NDS}(\bar{\Gamma}_G) = n(n-3)^2+2n.$$

□

3.3 Further Discussion

By inspection on the eigenvalues of the spectrum in Theorems 3.1 and 3.2 and taking the maximum of $|\lambda_i|$, then it is possible to derive the following two corollaries.

Corollary 3.1. *Let Γ_G be the non-commuting graph on G, where $G = D_{2n} \backslash Z(D_{2n})$, then $NDS-$spectral radius for Γ_G is*

$$\rho_{NDS}(\Gamma_G) = \begin{cases} \frac{-n^2}{2} + \frac{\sqrt{n^4+4n(n-1)(3n-2)^2}}{2}, & \text{if } n \text{ is odd} \\ \frac{-n^2}{2} + \frac{\sqrt{n^4+4n(n-2)(3n-4)^2}}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Corollary 3.2. *Let $\bar{\Gamma}_G$ be the commuting graph on G, where $G = D_{2n} \backslash Z(D_{2n})$, then the $NDS-$spectral radius for $\bar{\Gamma}_G$ is*

$$\rho_{NDS}(\bar{\Gamma}_G) = \begin{cases} n(n-2), & \text{if } n \text{ is odd} \\ (n-1)(n-3), & \text{if } n \text{ is even.} \end{cases}$$

It can be observed that the $NDS-$spectral radius of $\bar{\Gamma}_G$ for $G = D_{2n} \backslash Z(D_{2n})$ is always an even integer. While for Γ_G, it is never an odd integer.

Moreover, according to the results presented in the previous sections, the energies in Theorems 3.1 and 3.2 yield the following two corollaries:

Corollary 3.3. *Let $\bar{\Gamma}_G$ be the commuting graph on G, where $G = D_{2n} \backslash Z(D_{2n})$, then the $NDS-$energy for $\bar{\Gamma}_G$ is always an even integer.*

Corollary 3.4. *Let Γ_G be the non-commuting graph on G, where $G = D_{2n} \backslash Z(D_{2n})$, then $NDS-$energy for Γ_G is never an odd integer.*

The statements in Corollary 3.3 and 3.4 comply with the well known fact from [3] and [24] that the energy of a graph is never an odd integer as well as never the square root of an odd integer.

The following is an example of the neighbors degree sum energy of commuting and non-commuting graphs for D_{2n}, where $n = 4$.

Example 3.1. *Let $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ and $Z(D_8) = \{e, a^2\}$, where $C_{D_8}(a^i) = \{e, a, a^2, a^3\}$, $C_{D_8}(b) = \{e, a^2, b, a^2b\} = C_{D_8}(a^2b)$, $C_{D_8}(ab) = \{e, a^2, ab, a^3b\} = C_{D_8}(a^3b)$. For $G = D_8 \backslash Z(D_8)$, by using the information on the centralizer of each element in G, then Γ_G and $\bar{\Gamma}_G$ are as in Figure 1.*

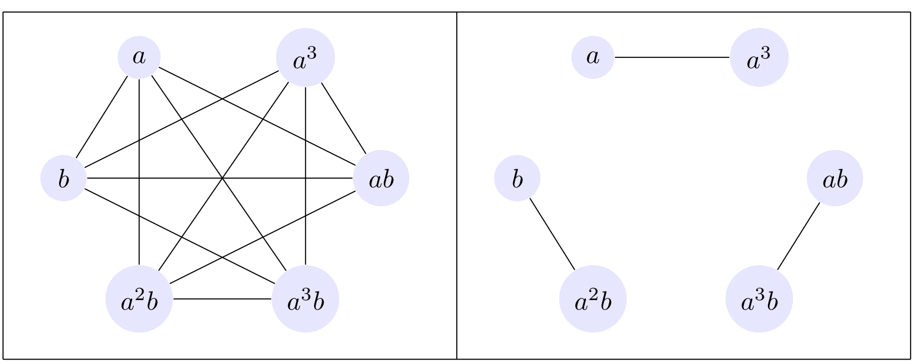

Figure 1: (i) Non-commuting graph on G, Γ_G; (ii) Commuting graph on G, $\bar{\Gamma}_G$

Now we construct 6×6 *neighbors degree sum matrices of* Γ_G *and* $\bar{\Gamma}_G$ *as the following:*

$$NDS(\Gamma_G) = \begin{array}{c} \\ a \\ a^3 \\ b \\ ab \\ a^2b \\ a^3b \end{array} \begin{array}{c} \begin{array}{cccccc} a & a^3 & b & ab & a^2b & a^3b \end{array} \\ \begin{pmatrix} -16 & 0 & 8 & 8 & 8 & 8 \\ 0 & -16 & 8 & 8 & 8 & 8 \\ 8 & 8 & -16 & 8 & 0 & 8 \\ 8 & 8 & 8 & -16 & 8 & 0 \\ 8 & 8 & 0 & 8 & -16 & 8 \\ 8 & 8 & 8 & 0 & 8 & -16 \end{pmatrix} \end{array}$$

$$= \begin{bmatrix} 16I_2 & 8J_2 & 8J_2 \\ 8J_2 & -(16+8)I_2 + 8J_2 & 8(J-I)_2 \\ 8J_2 & 8(J-I)_2 & -(16+8)I_2 + 8J_2 \end{bmatrix},$$

and

$$NDS(\bar{\Gamma}_G) = \begin{array}{c} \\ a \\ a^3 \\ b \\ ab \\ a^2b \\ a^3b \end{array} \begin{array}{c} \begin{array}{cccccc} a & a^3 & b & ab & a^2b & a^3b \end{array} \\ \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 \end{pmatrix} \end{array} = \begin{bmatrix} -(1+2)I_2 + 2J_2 & 0_2 & 0_2 \\ 0_2 & -I_2 & 2I_2 \\ 0_2 & 2I_2 & -I_2 \end{bmatrix}.$$

Here the characteristic polynomial of $NDS(\Gamma_G)$ *and* $NDS(\bar{\Gamma}_G)$ *are as follows:*

$$P_{NDS(\Gamma_G)}(\lambda) = (\lambda+16)^3(\lambda+32)^2(\lambda-16) \text{ and } P_{NDS(\bar{\Gamma}_G)}(\lambda) = (\lambda+3)^3(\lambda-1)^3.$$

By using Maple [7], *we have confirmed that*

$$Spec(\Gamma_G) = \left\{(16)^1, (-16)^3, (-32)^2\right\} \text{ and } Spec(\bar{\Gamma}_G) = \left\{(1)^3, (-3)^3\right\}.$$

Therefore, the $NDS-$*energy of* Γ_G *and* $\bar{\Gamma}_G$ *are as follows:*

$$\begin{aligned} E_{NDS}(\Gamma_G) =& (1)|16| + (3)|-16| + (2)|-32| = 128 \\ =& (4-3)^2 4^2 + 4 \cdot 4(4-2)^2 + \sqrt{4^4 + 4 \cdot 4(4-2)(3 \cdot 4 - 4)^2} \\ E_{NDS}(\bar{\Gamma}_G) =& (3)|1| + (3)|-3| = 12 = 4(4-3)^2 + 2(4). \end{aligned}$$

4 Conclusions

The energy formula of Γ_G and $\bar{\Gamma}_G$ for dihedral group D_{2n}, where $n \geq 3$, based on the $NDS-$eigenvalues, has been presented. The $NDS-$energy of Γ_G is either $(n-2)^2n^2+4n(n-1)^2+\sqrt{n^4+4n(n-1)(3n-2)^2}$, for odd n, or $(n-3)^2n^2+4n(n-2)^2+\sqrt{n^4+4n(3n-4)^2(n-2)}$, for even n. While the $NDS-$energy of $\bar{\Gamma}_G$ is either $(n+1)(n-2)^2$, for odd n, or $n(n-3)^2+2n$, for even n. It is found that the $NDS-$energy formulas we present here for both types of graphs, Γ_G and $\bar{\Gamma}_G$, are all aligned with previous literature which state that energy of graph is never an odd integer as well as never a square root of an odd integer.

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